

# The angular momentum of plane-fronted gravitational waves in the teleparallel equivalent of general relativity

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## Abstract

We present a simplified expression for the gravitational angular momentum in the framework of the teleparallel equivalent of general relativity (TEGR). The expression arises from the constraints equations of the Hamiltonian formulation of the TEGR. We apply this expression to the calculation of the angular momentum of plane-fronted gravitational waves in an arbitrary three-dimensional volume  $V$  of space and compare the results with those obtained for linearised gravitational waves.

Keywords: Gravitational waves, angular momentum, teleparallel gravity.

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## 1 Introduction

In the theory of general relativity, gravity is usually considered a phenomenon resulting from the curvature of space-time. The curvature of space-time is produced by the presence of matter, although there are vacuum solutions of the field equations. When massive objects move in space-time, the curvature changes as a consequence of the movement of the objects. Massive

objects that either acquire accelerated motion, or that undergo unexpected changes, generate disturbances in space-time that travel away from the object. In Einstein's theory these disturbances generate gravitational radiation and gravitational waves [1]. Although gravitational waves have not yet been detected directly, in principle they should carry energy, momentum and angular momentum as gravitational radiation [2, 4, 5]. Nowadays it is clear that gravity can also be described by tetrad fields and the torsion tensor, in the context of the teleparallel equivalent of general relativity [6]. The accelerated motion of massive bodies leads to changes in the space-time torsion as well.

The direct detection of gravitational waves will certainly open a new page in the history of general relativity. For example, by means of these waves, the theory of general relativity can be tested in the limit of strong gravitational fields, and not only as corrections to the Newtonian theory or tests based on linearised solutions of Einstein's equations. These waves carry important information about their source. In addition, the detection of these waves may provide insights about the properties of the waves such as velocity, helicity and polarization states[1].

The detection of gravitational waves requires an appropriate description of physically observable quantities associated with them, such as energy, momentum and angular momentum. As we know, in the realm of general relativity, there are no widely accepted definitions for these quantities, which are usually analysed by means of pseudotensors. Recently, in the context of linearised gravitational waves, the angular momentum of the waves has been investigated in the framework of pseudotensors [4]. However, an analysis of the gravitational angular momentum by means of pseudotensors is not satisfactory, because pseudotensors are coordinate dependent quantities.

In this work, we first recall that in the framework of the teleparallel equivalent of general relativity, the problems of energy, momentum and angular momentum of gravitational waves are issues that are well established. The expressions for these quantities arise from the integral form of the constraint equations of the Hamiltonian formulation of theTEGR. The resulting expressions are invariant under coordinate transformations of the three-dimensional space, and under time reparametrizations. Here we apply the newly achieved, simplified definition of the gravitational angular momentum to plane-fronted gravitational waves in an arbitrary three-dimensional volume  $V$  of space. We also present the non-zero components of the energy-momentum vector of the waves, and compare the results obtained for plane-fronted gravitational

waves with those obtained for linearised gravitational waves.

In order to check the consistency of the results obtained here, we analyze the behaviour of a particle of mass  $m$  in the presence of gravitational waves discussed in this work, in order to analyse how the wave modifies the kinematic state of the particle by means of transfer of energy.

We use the following notation: spacetime indices  $\mu, \nu, \dots$  and  $SO(3, 1)$  indices  $a, b, \dots$  run from 0 to 3. Time and space indices are indicated according to  $\mu = 0, i$ ,  $a = (0), (i)$ . The tetrad fields are indicated by  $e^a{}_\mu$ , and the flat Minkowski spacetime metric tensor  $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = (-1, 1, 1, 1)$  raises and lowers tetrad indices. The determinant of tetrad fields is represented by  $e = \det(e^a{}_\mu) = \sqrt{-g}$  and we use the constants  $G = c = 1$ .

## 2 The Lagrangian and Hamiltonian formulations of the TEGR

In this section we present a summary of both the Lagrangian and Hamiltonian formulations of the TEGR. For more details see Refs. [6, 7, 8, 9, 10, 12]. The equivalence of the TEGR with Einstein's general relativity is obtained by means of an identity that relates the scalar curvature  $R(e)$  constructed out of the tetrad fields and a combination of quadratic terms in the torsion tensor [6, 7, 8, 9, 10],

$$eR(e) \equiv -e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) + 2\partial_\mu (eT^\mu), \quad (1)$$

where  $e = \det(e^a{}_\mu)$ ,  $T_a = T^b{}_{ba}$ ,  $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$  and  $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$  is the torsion tensor. The Lagrangian density in the TEGR is given by

$$\begin{aligned} L(e) &= -e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - L_M \\ &\equiv -ke \Sigma^{abc} T_{abc} - L_M, \end{aligned} \quad (2)$$

where  $k = 1/(16\pi)$ ,  $L_M$  represent the Lagrangian density for the matter fields, and  $\Sigma^{abc}$  is defined by

$$\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c). \quad (3)$$

The quadratic combination,  $\Sigma^{abc} T_{abc}$ , except for a total divergence, is proportional to the scalar curvature  $R(e)$ .

The variation of  $L(e)$  with respect to  $e^{a\mu}$  yields the fields equations which, after some rearrangements, may be written in the form

$$\partial_\nu(e\Sigma^{a\mu\nu}) = \frac{1}{4k}ee^a{}_\nu(t^{\mu\nu} + T^{\mu\nu}), \quad (4)$$

where

$$t^{\mu\nu} = k(4\Sigma^{bc\mu}T_{bc}{}^\nu - g^{\mu\nu}\Sigma^{bcd}T_{bcd}), \quad (5)$$

is interpreted as the gravitational energy-momentum tensor [6] and  $T^{\mu\nu} = e_a{}^\mu T^{a\nu}$ , where  $eT_{a\mu} = \delta L_M / \delta e^{a\mu}$ . These field equations are equivalent to Einstein's equations. It is possible to verify that the equations can be written as  $\frac{1}{2}(R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e))$ , provided we make  $L_M = 0$ .

In order to obtain the Hamiltonian density in the TEGR we rewrite the Lagrangian density  $L$  in the phase space. To do this, first we note that the Lagrangian density (2) does not depend on the time derivative of the tetrad component  $e_{a0}$ . Therefore this component appears as a Lagrange multiplier in the Hamiltonian density  $H$ . From (2) we can obtain the momentum canonically conjugated to  $e_{ai}$  as  $\Pi^{ai} = \delta L / \delta \dot{e}_{ai} = -4k\Sigma^{a0i}$ , and  $\Pi^{a0} \equiv 0$ . The Hamiltonian density is obtained by rewriting the Lagrangian density in the form  $L = \Pi^{ai}\dot{e}_{ai} - H$ , in terms of  $e_{ai}$ ,  $\Pi^{ai}$  and Lagrange multipliers. After the Legendre transform is performed, we obtain the final form of the Hamiltonian density, that is given by [8]

$$H(e, \Pi) = e_{a0}C^a + \lambda_{ab}\Gamma^{ab}. \quad (6)$$

In the above equation we have omitted a surface term.  $C^a = \delta H / \delta e_{a0}$  is a very long expression of the field variables, and  $\Gamma^{ab}$  is defined by  $\Gamma^{ab} = 2\Pi^{[ab]} + 4ke(\Sigma^{a0b} - \Sigma^{b0a})$ . In  $H$ ,  $e_{a0}$  and  $\lambda_{ab}$  are Lagrange multipliers which, after solving the field equations, are identified as  $\lambda_{ab} = (1/4)(T_{a0b} - T_{b0a} + e_a{}^0 T_{00b} - e_b{}^0 T_{00a})$ . The quantities  $C^a$  and  $\Gamma^{ab}$ , as functions of  $\Pi^{ai}$  and  $e_{ai}$ , are first class constraints [8]. It is possible to show that, in terms of the Poisson brackets, these constraints satisfy an algebra similar to the algebra of the Poincaré group [8].

The form of the constraints  $C^a$  is given by

$$C^a = -\partial_i \Pi^{ai} + h^a = 0. \quad (7)$$

Where  $h^a$  is a intricate expression of the field quantities.

The relevant result presented in this work is that the constraint  $\Gamma^{ab}$  can be simplified and rewritten as a total divergence according to

$$\Gamma^{ab} = 2\Pi^{[ab]} - 2k\partial_i[e(e^{ai}e^{b0} - e^{bi}e^{a0})] = 0. \quad (8)$$

Two important results of the TEGR are the following: both constraints  $C^a$  and  $\Gamma^{ab}$  may be written in terms of a total divergence which, under integration, yield straightforward expressions for the gravitational energy-momentum and angular momentum, respectively. In particular, the integral form of the constraint  $C^a = 0$  yields the gravitational energy-momentum  $P^a$  [13],

$$P^a = - \int_V d^3x \partial_i \Pi^{ai}, \quad (9)$$

where  $V$  is an arbitrary volume of the three-dimensional space and  $\Pi^{ai} = -4k\Sigma^{a0i}$ .

In Ref. [13], motivated by the definition of  $P^a$  as an integral of a total divergence, we presented an expression for the angular momentum of the gravitational field out of the integral form of the constraint  $\Gamma^{ij} = 2\Pi^{[ij]} = e_a{}^i \Gamma^{ab} e_b{}^j = 0$ , where  $2\Pi^{[ij]}$  was taken as the gravitational angular momentum density. In Ref. [14], by writing the Hamiltonian  $H$  in a more simple form, we redefined the angular momentum of the gravitational field as the integral of the constraint  $\Gamma^{ab} = 0$ . Unlike the expressions presented in [13, 14], here we note that, after some simplifications, the constraint  $\Gamma^{ab} = 0$  given by Eq. (8) may be written in terms of a total divergence.

In similarity with the definition given in [14], we define the 4-angular momentum of the gravitational field as

$$L^{ab} = - \int_V d^3x M^{ab}, \quad (10)$$

where

$$M^{ab} = 2\Pi^{[ab]} = (\Pi^{ab} - \Pi^{ba}) = 2k\partial_i[e(e^{ai}e^{b0} - e^{bi}e^{a0})]. \quad (11)$$

Expressions (9) and (10) are both invariants under coordinate transformations of the three-dimensional space, and under time reparametrizations. Note that the integrals in (9) and (10) may be carried out on surfaces that enclose an arbitrary volume  $V$ . The expression above generalizes the one obtained in [15]. The latter was also given as a total divergence, but the result was obtained for the metric tensor with axial symmetry..

It is possible to show, using Poisson brackets, that the quantities  $P^a$  and  $L^{ab}$  defined in (9) and (10) respectively, satisfy the algebra of the Poincaré group,

$$\{P^a, P^b\} = 0, \quad (12)$$

$$\{P^a, L^{bc}\} = \eta^{ab}P^c - \eta^{ac}P^b, \quad (13)$$

$$\{L^{ab}, L^{cd}\} = \eta^{ad}L^{cb} + \eta^{bd}L^{ac} - \eta^{ac}L^{db} - \eta^{bc}L^{ad}. \quad (14)$$

Therefore, from a physical point of view, the interpretation of the quantities  $P^a$  and  $L^{ab}$  is consistent.

In the next section we will apply the definition of  $L^{ab}$  to the calculation of the components of the angular momentum carried by a plane-fronted gravitational wave. We will also present the non-zero components of the energy-momentum vector calculated in Ref. [3], and compare these results with those obtained for linearised gravitational waves.

### 3 The angular momentum of gravitational waves

A plane-fronted gravitational wave propagating in the  $z$  direction may be described by the line element [16]

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x, y, u)du^2. \quad (15)$$

The vacuum field equations are reduced to

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) H(x, y, u) = 0. \quad (16)$$

By writing the coordinates  $(u, v)$  in terms of the coordinates  $(t, z)$ ,

$$u = \frac{1}{\sqrt{2}}(z - t), \quad v = \frac{1}{\sqrt{2}}(z + t),$$

the line element in (15) becomes

$$ds^2 = \left( \frac{H}{2} - 1 \right) dt^2 + dx^2 + dy^2 + \left( \frac{H}{2} + 1 \right) dz^2 - H dt dz. \quad (17)$$

In the following calculations we need the inverse of the metric tensor. It is given by

$$g^{\mu\nu} = \begin{pmatrix} -1 - \frac{H}{2} & 0 & 0 & -\frac{H}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{H}{2} & 0 & 0 & 1 - \frac{H}{2} \end{pmatrix}. \quad (18)$$

In order to evaluate  $L^{ab}$  associated with the gravitational wave described by (17), we consider a set of tetrad fields adapted to static observers. These tetrad fields must satisfy  $e_{(0)}^i = 0$ , and are given by

$$e_{a\mu} = \begin{pmatrix} -A & 0 & 0 & -B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \quad (19)$$

where

$$A = \left(1 - \frac{H}{2}\right)^{1/2}, \quad AB = \frac{H}{2}, \quad AC = 1.$$

The tetrads in (19) are adapted to static observers. They are better understood if we consider the inverse  $e_a{}^\mu$ . In terms of components, we have

$$e_{(0)}{}^\mu = (1/A, 0, 0, 0), \quad (20)$$

and

$$e_{(1)}{}^\mu = (0, 1, 0, 0), \quad e_{(2)}{}^\mu = (0, 0, 1, 0), \quad e_{(3)}{}^\mu = (-H/(2A), 0, 0, A). \quad (21)$$

The four velocity of the frame is given by  $u^\mu = e_{(0)}{}^\mu$ . Therefore, equation (20) fixes the kinematic state of the frame: the three velocity conditions  $e_{(0)}^i = 0$ , ensure that the frame is static. The other three conditions in (21) fix the spatial orientation of the frame, i.e.,  $e_{(1)}{}^\mu, e_{(2)}{}^\mu$  and  $e_{(3)}{}^\mu$  are unit vectors along the  $x, y$  and  $z$  directions, respectively. An alternative way to characterize the tetrad fields is by fixing the six components of the acceleration tensor  $\phi_{ab} = -\phi_{ba} = (1/2)(T_{(0)ab} + T_{a(0)b} - T_{b(0)a})$  [17], where  $\phi_{a(0)}$  and  $\phi_{(i)(j)}$  are translational and rotational accelerations, respectively. These accelerations are necessary to maintain the frame in a given inertial state in spacetime. Here we note that if we take  $H = 0$ ,  $e_a{}^\mu = \delta_a^\mu$ , and therefore  $T_{a\mu\nu} = 0$ .

Although the components of  $M^{ab}$  do not depend on the torsion tensor explicitly, the latter is important to calculate the energy-momentum vector  $P^a$  and the acceleration tensor  $\phi_{ab}$ . For the tetrad fields given by (19), the non-vanishing components  $T_{\mu\nu\lambda}$  were calculated in Ref. [?]. In the latter reference we addressed the non-linear gravitational wave given by Eq. (15), and the same tetrad fields described by Eq. (19). We showed that the torsion tensor obtained in the framework of these gravitational waves breaks parallelograms in space-time, and obtained a physical result that cannot be achieved in the realm of the standard formulation of general relativity. We will return to this point in the final remarks of this article.

In order to calculate the components of the angular momentum transported by the waves described by (17), we need the quantities given by (20) and (21). The components of the angular momentum can be obtained as surface integrals. However, since we do not dispose of the explicit form of  $H$ , we find it more appropriate to calculate these components as volume integrals, according to

$$L^{ab} = -2k \int_V \partial_j [e(e^{aj}e^{b0} - e^{a0}e^{bj})] d^3x. \quad (22)$$

The determinant of tetrad fields is  $e = AC = 1$ . From (20), (21) and (22) we obtain the nonvanishing components of  $L^{ab}$ . They read

$$L^{(0)(1)} = -2k \int_V \partial_x \left( \frac{1}{A} \right) d^3x, \quad (23)$$

$$L^{(0)(2)} = -2k \int_V \partial_y \left( \frac{1}{A} \right) d^3x, \quad (24)$$

$$L^{(1)(3)} = 2k \int_V \partial_x \left( \frac{H}{2A} \right) d^3x, \quad (25)$$

$$L^{(2)(3)} = 2k \int_V \partial_y \left( \frac{H}{2A} \right) d^3x. \quad (26)$$

We observe that the plane-fronted gravitational waves do not carry angular momentum in the direction of propagation, i.e.,  $L^z = L^{(1)(2)} = 0$ , but only in the orthogonal direction.

The non-vanishing components of the gravitational energy-momentum vector  $P^a$  of plane-fronted gravitational waves have been calculated in Ref. [3]. However, we present them here to compare their values with those obtained for linearised gravitational waves. The non-zero expressions of  $P^a$  given by (9), constructed out of Eq. (19), read



$$P^{(0)} = P^{(3)} = -\frac{k}{8} \int_V \left[ \frac{1}{A^3} [(\partial_x H)^2 + (\partial_y H)^2] \right] d^3x. \quad (27)$$

We note that  $P^2 = \eta_{ab} P^a P^b = 0$ . This result is consistent with the fact that the gravitational waves should describe massless particles. We will now compare this result with that obtained in the context of a linearised gravitational wave propagating in the  $z$  direction.

In the diagonal polarization, the linearised gravitational wave is described by the line element,

$$ds^2 = -dt^2 + [1 + f(t - z)]dx^2 + [1 - f(t - z)]dy^2 + dz^2, \quad (28)$$

where  $|f(t - z)| \ll 1$  and  $e = \sqrt{-g} = 1$ . The tetrad fields adapted to static observers, in the space-time determined by (28), is given by

$$e_{a\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + \frac{f}{2} & 0 & 0 \\ 0 & 0 & 1 - \frac{f}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (29)$$

The non-vanishing components of the torsion tensor  $T_{a\mu\nu}$  are

$$T_{(1)01} = \frac{1}{2}\partial_t f, \quad T_{(2)02} = -\frac{1}{2}\partial_t f, \quad T_{(1)13} = -\frac{1}{2}\partial_z f, \quad T_{(2)23} = \frac{1}{2}\partial_z f. \quad (30)$$

We notice that if we consider only first order terms in  $f$ , all components of the acceleration tensor  $\phi_{ab}$  vanish. This fact means, as we will see in the next section, that linearised gravitational waves do not transfer energy to a particle of mass  $m$ . Since the inverse of the tetrad fields above is also represented by a diagonal matrix, it is not difficult to conclude from Eq. (22) that the components of the angular momentum carried by linearised gravitational waves are given by

$$L^{(0)(i)} = 2k \int_V \partial_j (e e^{(i)j}) d^3x = 0, \quad L^{(i)(k)} = 0, \quad (31)$$

where in the first equation above we use the fact that  $e = 1$ , and  $e^{(1)1} = 1 - f/2$  and  $e^{(2)2} = 1 + f/2$  are functions only of  $t - z$ . This result implies that in the framework of the TEGR, gravitational waves in the linearised approximation do not transport angular momentum. In the linear approximation, and in the context of the tetrad fields (29), we can easily show that

all the components of  $\Sigma^{a0j}$  vanish. Therefore, all components of  $P^a$  given by Eq. (9) are also zero. Thus, unlike plane-fronted gravitational waves, gravitational waves in the linearised approximation do not carry energy, momentum and angular momentum.

## 4 Polarization of plane-fronted gravitational waves

The polarization of plane-fronted gravitational waves may be obtained from the analysis of the acceleration tensor  $\phi_{ab}$ , and the behaviour of a free particle in the gravitational field. We recall that  $\phi_{ab}$  represent the inertial (i.e., non-gravitational) accelerations that are necessary to maintain the frame in a given inertial state. In the present context, we are considering a static frame in space-time, defined by Eqs. (20) and (21). The values of  $\phi_{ab}$  that follow from Eq. (19) and the latter equations are given by

$$\phi_{(0)(1)} = -\frac{1}{4A^2}\partial_x H, \quad (32)$$

$$\phi_{(0)(2)} = -\frac{1}{4A^2}\partial_y H, \quad (33)$$

$$\phi_{(0)(3)} = \frac{1}{4A^3}\partial_z H, \quad (34)$$

$$\phi_{(1)(2)} = 0, \quad (35)$$

$$\phi_{(1)(3)} = -\frac{1}{4A^2}\partial_x H, \quad (36)$$

$$\phi_{(2)(3)} = -\frac{1}{4A^2}\partial_y H. \quad (37)$$

If we imagine a free particle in space-time that is *not* subject to the inertial conditions established by Eqs. (19-21), then the presence of the wave will impart the *negative* values of the accelerations above on the particle. Equations (32) and (33) imply that the wave is transversal, since the particle will move in the transversal  $x$  and  $y$  directions (and possibly oscillate, depending on the form of the function  $H$ ). Equation (34) implies a longitudinal motion.

The resulting motion of the free particle is, to some extent, similar to the behaviour of an electron under the action of an intense laser beam, in which case we know that the motion of the electron results in the well known “figure 8”, that generates the Thomson scattering. Therefore, the plane-fronted gravitational wave is both a transversal and longitudinal wave.

## 5 Massive particle in the presence of gravitational waves

In order to verify the consistency of the results obtained up to here, let us analyse the transfer of energy to a particle of mass  $m$  in the presence of gravitational waves. First, from the point of view of the Euler-Lagrange equations, we will consider the particle in the presence of a plane-fronted gravitational wave, and then, in a similar way, we will consider the particle in the presence of a linearised gravitational wave.

For the particle in the presence of a plane-fronted gravitational wave described by (17), the Lagrangian  $L = \frac{m}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$  that describes the particle motion is written as

$$L = \frac{m}{2} \left[ (H/2 - 1)\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + (H/2 + 1)\dot{z}^2 - H\dot{t}\dot{z} \right], \quad (38)$$

where  $\dot{x}^\mu = dx^\mu/d\tau$ , with  $\tau$  being the proper time of the particle. The Euler-Lagrange equations for the  $t, x, y, z$  coordinates are, respectively

$$2\ddot{t} + \sqrt{2}H\ddot{u} + \sqrt{2}\dot{H}\dot{u} - \frac{1}{\sqrt{2}}\frac{\partial H}{\partial u}\dot{u}^2 = 0, \quad (39)$$

$$2\ddot{x} - \frac{\partial H}{\partial u}\dot{u}^2 = 0, \quad (40)$$

$$2\ddot{y} - \frac{\partial H}{\partial u}\dot{u}^2 = 0, \quad (41)$$

$$2\ddot{z} + \sqrt{2}H\ddot{u} + \sqrt{2}\dot{H}\dot{u} - \frac{1}{\sqrt{2}}\frac{\partial H}{\partial u}\dot{u}^2 = 0. \quad (42)$$

From the first and fourth equations above we get

$$\ddot{z} - \ddot{t} = 0 \rightarrow \ddot{u} = 0 \rightarrow \dot{u} = \frac{1}{\sqrt{2}}(\dot{t} - \dot{z}) \equiv k = \text{constant}. \quad (43)$$

We recall that the tetrad fields (19) are interpreted as a reference frame adapted to static observers in space-time, since we identify the  $e_{(0)}{}^\mu$  components of the frame with the four-velocities  $u^\mu$  of the observers,  $e_{(0)}{}^\mu = u^\mu = (1/A, 0, 0, 0)$ . Thus, if we consider the particle of mass  $m$  in the presence of plane-fronted gravitational waves, its four-momentum, when measured by an observer adapted to (19), will be given by  $p_a = e_a{}^\mu p_\mu$ , where  $p^\mu = m\dot{x}^\mu$ . Therefore, the energy of the particle measured by observers adapted to (19) is given by

$$E = -p_\mu u^\mu = -e_{(0)}{}^\mu p_\mu = -e_{(0)}{}^0 p_0 = -\frac{p_0}{A}, \quad (44)$$

where  $p_0 = g_{0\mu} p^\mu = g_{00} \dot{t} + g_{03} \dot{z}$ . Substituting  $p_0$  in the equation above and expressing  $A$ ,  $g_{00}$  and  $g_{03}$  in terms of  $H$ , we are left with

$$E = m[(1 - H/2)^{1/2} \dot{t} + (H/2)(1 - H/2)^{-1/2} \dot{z}]. \quad (45)$$

From the line element (17) it follows that

$$d\tau^2 = (-g_{00})dt^2 \rightarrow \dot{t} = \frac{dt}{d\tau} = \left(1 - \frac{H}{2}\right)^{-1/2}. \quad (46)$$

Using (45) and the equation above, we can eliminate  $\dot{z}$  and  $\dot{t}$  in terms of the constant  $k$  and the function  $H$ , so that we may rewrite  $E$  as

$$E = m \left(1 + \frac{H/2}{1 - H/2}\right) + m\sqrt{2}k \left(\frac{H/2}{1 - H/2}\right). \quad (47)$$

Here we note that the plane-fronted gravitational wave changes the kinematic state of the particle by means of energy transfer. In particular, if we take  $k = 0$ , and if  $H < 0$ , the energy of the particle in the presence of the wave will be lower than its rest energy, i. e.  $E < E_0 = m$ .

Let us now consider the particle of mass  $m$  in the presence of a linearised gravitational wave. For the line element in (28), the Lagrangian is

$$L = \frac{m}{2} \left(-\dot{t}^2 + g_{11}\dot{x}^2 + g_{22}\dot{y}^2 + \dot{z}^2\right), \quad (48)$$

and the Euler-Lagrange equations in the coordinates  $t, x, y, z$  reduce to

$$2\ddot{t} + \frac{\partial g_{11}}{\partial u}\dot{x}^2 + \frac{\partial g_{22}}{\partial u}\dot{y}^2 = 0, \quad (49)$$

$$\frac{d}{d\tau}(g_{11}\dot{x}) = 0, \quad (50)$$

$$\frac{d}{d\tau}(g_{22}\dot{y}) = 0, \quad (51)$$

$$2\ddot{z} + \frac{\partial g_{11}}{\partial u}\dot{x}^2 + \frac{\partial g_{22}}{\partial u}\dot{y}^2 = 0, \quad (52)$$

Again, from the first and fourth equations above we have  $\ddot{t} - \ddot{z} = 0 \rightarrow \dot{t} - \dot{z} = k' = \text{constant}$ .

The four-velocity  $u^\mu$  of observers adapted to the tetrad fields given by (29) is  $u^\mu = e_{(0)}^\mu = (1, 0, 0, 0)$ . Therefore, the energy of the particle in the presence of a linearised gravitational wave, measured by observers adapted to (29), is  $E = -p_\mu u^\mu = -p_0 = p^0 = m(dt/d\tau)$ . Since  $dt/d\tau = 1/(-g_{00})^{1/2} = 1$ , the energy of the particle in the presence of a linearised gravitational wave is simply

$$E = m. \quad (53)$$

So, from the point of view of static observers, the linearised gravitational wave does not transfer energy to the particle. This result is consistent with the previously obtained conclusion in Section 3, namely, that linearised gravitational waves do not transport energy.

## 6 Final remarks

In this work we have presented a very simple expression for the density of angular momentum of the gravitational field. The expression, given by Eq. (11), arises in the constraint equations of the Hamiltonian formulation of the TEGR and is given by a total divergence. The definition in question is invariant under coordinate transformations of the three-dimensional space, and under time reparametrizations. In addition, the definitions of energy-momentum and angular momentum presented in equations (9) and (10) are both invariant under global  $S(3,1)$  transformations of the frame. However, we must remember that in the theory of special relativity, physical quantities such as energy and momentum are frame dependent, and we believe that a similar property is expected to hold in the theory of general relativity. Moreover, in view of Eq. (10), we see that the gravitational angular momentum enclosed by a three-dimensional volume  $V$  can be calculated for arbitrary values of the volume  $V$ .

We applied the definition given by Eq. (10), and calculated the components of the angular momentum within a volume  $V$ , carried by (i) a plane-fronted gravitational wave, and by (ii) a linearised gravitational wave. All calculations were performed with respect to an observer adapted to a static reference frame. Unlike the case for plane-fronted gravitational waves, the angular momentum carried by linearised gravitational wave vanishes. For the plane-fronted gravitational wave, the component of the angular momentum in the direction of propagation of the wave vanishes. From the point of view of the TEGR, we show that, contrary to what happens in the case of plane-fronted gravitational waves, linearised gravitational waves do not carry neither energy nor angular momentum.

In order to better understand the results obtained in the present context, we have reviewed in section 5 the change of the energy of a particle of mass  $m$ , in a frame adapted to static observers, in the presence of gravitational waves. Equation (47) shows that in the presence of a plane-fronted gravitational wave, the energy of the particle is modified, and energy transfer may occur between the particle and the wave. Therefore, we conclude that in this case the wave carries energy. When we consider the particle in the presence of a linearised gravitational wave, we see from Eq. (53) that the energy of the particle adapted to a static reference frame is not modified. Therefore we conclude that linearised gravitational waves do not transfer energy to the particle.

The equivalence of the TEGR with the standard formulation of general relativity holds at the level of field equations. However, in the TEGR there are physical predictions that cannot be achieved in the standard, metrical formulation of the theory. In Ref. [?] we have considered an experimental setup similar to the two perpendicular arms of the interferometer constructed at the LIGO experiment. Likewise, we assumed that a laser beam travels back and forth along the two arms, at the end of which there are *fixed* mirrors. Considering that the two perpendicular arms are established in the  $xy$  plane, say, we found that in the presence of a plane fronted gravitational wave that travels in the  $z$  direction there is a lapse of time between the departure (at the same point in the three-dimensional space) and arrival of the two laser beams, along the perpendicular directions. This lapse of time is a consequence of the breaking of a certain parallelogram in space-time [?], and cannot be obtained in the realm of the metrical formulation of general relativity, since it is described by the space-time torsion tensor. This is an interesting, distinctive feature that (i) cannot be established in the context of

linearised gravitational waves, (ii) is feasible of being measured, and (iii) that cannot be explained in terms of the curvature tensor. In the TEGR we deal with tetrad fields, which, in a certain sense, are the “square root” of metric tensor. Thus, it is natural that some physical predictions of the TEGR do not have the corresponding counterpart in the metrical formulation.

## References

- [1] Barry C. Barish and Rainier Weiss, *Physics Today* **52**, (10) 44 (1999).
- [2] T. Dereli and R. W. Tucker, *Classical Quantum Gravity* **21**, 1459 (2004).
- [3] J. W. Maluf and S. C. Ulhoa, *Phys. Rev. D* **78**, 047502 (2008).
- [4] L. M. Butcher, A. Lasenby and M. Hobson, *Phys. Rev. D* **86** 084013 (2012).
- [5] L. M. Butcher, A. Lasenby and M. Hobson, *Phys. Rev. D* **86** 084012 (2012).
- [6] J. W. Maluf, *Ann. Phys. (Berlin)*, **525**, 339 (2013).
- [7] J. W. Maluf and J. F. da Rocha-Neto, *Phys. Rev. D* **64** 084014 (2001).
- [8] J. F. da Rocha-Neto, J. W. Maluf and S. C. Ulhoa, *Phys. Rev. D* **82** 124035 (2010).
- [9] F. W Hehl, *in Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity, Erice, 1979*, edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1980).
- [10] F. W Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman *Phys. Rep.* **258**, 1 (1995).
- [11] M. Blagojevic, *Gravitation and Gauge Symmetries* (IOP. Bristol 2002).
- [12] T. Ortin, *Gravity and Strings* (Cambridge Univ. Press. Cambridge, 2004).
- [13] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toríbio and K. H. Castello-Branco, *Phys. Rev. D* **65** 124001 (2002).

- [14] J. W. Maluf, S. C. Ulhoa, F. F. Faria and J. F. da Rocha Neto, Class. Quantum Grav. **23** 6245 (2006).
- [15] J. W. Maluf and S. C. Ulhoa, Gen. Rel. Gravit. **41**, 1233 (2009).
- [16] D. Kramer, H. Stephani, M. A. H. MacCollum and H. Herlt; *Exact Solutions of the Einstein's Fields Equations*, Cambridge University Press, Cambridge, (1980).
- [17] J. W. Maluf, F. F. Faria and S. C. Ulhoa, Class. Quantum Grav. **24** 2743 (2007).